CURVATURE AS A DIFFERENTIAL 2-FORM WITH VALUES IN THE ENDOMORPHISMS OF A VECTOR BUNDLE

Problem:
(from a lecture on Differential Geometry by K. Mohnke / N. Roy)

In an abstract setting, a generalization of Riemannian Curvature can be understood as a differential 2-Form with values in the endomorphisms of a vector bundle: Let ∇^Ebe a covariant derivative on a vector bundle E over a manifold M and let $F \in \Omega^2(M, end(E))$ be its curvature form. ∇^E induces a covariant derivative $\nabla^{end(E)}$ on the vector bundle $end(E)$. This induces an exterior covariant derivative

$$D^{end} : \Omega^k(M, end(E)) \to \Omega^{k+1}(M, end(E)).$$

Prove the *second Bianchi identity* $D^{end}F = 0$ by using Cartan's formula and

$$F(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma$$

Solution:

As a first step, apply Cartan's formula to $(D^{end}F(X,Y,Z))$:

$$\begin{aligned}
(D^{end}F(X,Y,Z))\sigma &= (\nabla_X^{end}F(Y,Z) - \nabla_Y^{end}F(X,Z) + \nabla_Z^{end}F(X,Y) \\
&\quad -F([X,Y],Z) + F([X,Z],Y) - F([Y,Z],X))\sigma \\
&= \nabla_X^{end}F(Y,Z)\sigma - \nabla_Y^{end}F(X,Z)\sigma + \nabla_Z^{end}F(X,Y)\sigma \\
&\quad -F([X,Y],Z)\sigma + F([X,Z],Y)\sigma - F([Y,Z],X)\sigma
\end{aligned}$$

Then recall the "characteristic property" of the covariant derivative on $end(E)$:

$$\nabla_X^E(F(Y,Z)\sigma) \quad = \quad (\nabla_X^{end}F(Y,Z))\sigma + F(Y,Z)\nabla_X^E\sigma,$$

thus

$$(\nabla_X^{end}F(Y,Z))\sigma = \nabla_X^E(F(Y,Z)\sigma) - F(Y,Z)\nabla_X^E\sigma$$

This leads to:

$$\begin{aligned}
(D^{end}F(X,Y,Z))\sigma &= (\nabla_X^E(F(Y,Z)\sigma) - F(Y,Z)\nabla_X^E\sigma) - (\nabla_Y^E(F(X,Z)\sigma) - F(X,Z)\nabla_Y^E\sigma) \\
&\quad +(\nabla_Z^E(F(X,Y)\sigma) - F(X,Y)\nabla_Z^E\sigma) \\
&\quad -F([X,Y],Z)\sigma + F([X,Z],Y)\sigma - F([Y,Z],X)\sigma
\end{aligned}$$

Now apply formula (1) several times:

$$
\begin{aligned}
(D^{end}F(X,Y,Z))\sigma &= \nabla^E_X(\nabla^E_Y\nabla^E_Z\sigma - \nabla^E_Z\nabla^E_Y\sigma - \nabla^E_{[Y,Z]}\sigma) - F(Y,Z)\nabla^E_X\sigma \\
&\quad -\nabla^E_Y(\nabla^E_X\nabla^E_Z\sigma - \nabla^E_Z\nabla^E_X\sigma - \nabla^E_{[X,Z]}\sigma) + F(X,Z)\nabla^E_Y\sigma \\
&\quad +\nabla^E_Z(\nabla^E_X\nabla^E_Y\sigma - \nabla^E_Y\nabla^E_X\sigma - \nabla^E_{[X,Y]}\sigma) - F(X,Y)\nabla^E_Z\sigma \\
&\quad -\nabla^E_{[X,Y]}\nabla^E_Z\sigma + \nabla^E_Z\nabla^E_{[X,Y]}\sigma + \nabla^E_{[[X,Y],Z]}\sigma \\
&\quad +\nabla^E_{[X,Z]}\nabla^E_Y\sigma - \nabla^E_Y\nabla^E_{[X,Z]}\sigma - \nabla^E_{[[X,Z],Y]}\sigma \\
&\quad -\nabla^E_{[Y,Z]}\nabla^E_X\sigma + \nabla^E_X\nabla^E_{[Y,Z]}\sigma + \nabla^E_{[[Y,Z],X]}\sigma \\
&= 0,
\end{aligned}
$$

when observing that one can use the "Jacobi-identity"

$$
\begin{aligned}
\nabla^E_{[[X,Y],Z]}\sigma - \nabla^E_{[[X,Z],Y]}\sigma + \nabla^E_{[[Y,Z],X]}\sigma &= \\
\nabla^E_{[[X,Y],Z]}\sigma + \nabla^E_{[[Z,X],Y]}\sigma + \nabla^E_{[[Y,Z],X]}\sigma &= 0
\end{aligned}
$$

and realizing that everything else cancels also, because

- $$\nabla^E_X\nabla^E_Y\nabla^E_Z\sigma - \nabla^E_Y\nabla^E_X\nabla^E_Z\sigma - \nabla^E_{[X,Y]}\nabla^E_Z\sigma = F(X,Y)\nabla^E_Z\sigma$$
- $$-\nabla^E_X\nabla^E_Z\nabla^E_Y\sigma + \nabla^E_Z\nabla^E_X\nabla^E_Y\sigma + \nabla^E_{[X,Z]}\nabla^E_Y\sigma = -F(X,Z)\nabla^E_Y\sigma$$
- $$\nabla^E_Y\nabla^E_Z\nabla^E_X\sigma - \nabla^E_Z\nabla^E_Y\nabla^E_X\sigma - \nabla^E_{[Y,Z]}\nabla^E_Y\sigma = F(Y,Z)\nabla^E_X\sigma$$

This finishes the proof of $D^{end}F = 0$.