C^1- Stable - Manifolds for Periodic Heteroclinic Chains in Bianchi IX

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Abstract

In this paper we study oscillatory Bianchi models of class A and are able to show that for admissible periodic heteroclinic chains in Bianchi IX there exisist C^1- stable - manifolds of orbits that follow these chains towards the big bang. A detailed study of Takens Linearization Theorem and the Non-Resonance-Conditions leads us to this new result in Bianchi class A. More precisely, we can show that there are no heteroclinic chains in Bianchi IX with constant continued fraction development that allow Takens-Linearization at all of their base points. Geometrically speaking, this excludes "symmetric" heteroclinic chains with the same number of "bounces" near all of the 3 Taub Points - the result shows that we have to require some "asymmetry" in the bounces in order to allow for Takens Linearization, e.g. by considering admissible 2-periodic continued fraction developments.

Contents

1. Resonances for Periodic Chains in Bianchi IX

We are interested in periodic heteroclinic chains in the Bianchi IX cosmological model. These can be represented by a Kasner parameter $u \in \mathbb{R}$ with an infinite periodic continued fraction representation, e.g. $u = [a, b, c, a, b, c, ...]$.

1.1. Infinite Periodic Continued Fractions.

From the theory of continued fractions, we know that it holds:

THEOREM. *$u \in \mathbb{R}$ has an infinite periodic continued fraction representation $\iff$ $u \in \mathbb{R}$ is a "quadratic irrational" $\iff$ u is a real but irrational root of a quadratic equation with integer coefficients, i.e. $\exists : c_1, c_2, c_3 : c_1 + c_2 u + c_3 u^2 = 0$ (with $c_i \in \mathbb{Z}$).*

Here we are only interested in the direction "$\Rightarrow$", which follows directly for the general formulas for continued fractions in section 3.3, see below. The other direction is a bit more elaborate (see e.g. **[37]** §19 or **[25]** §10).

For the argument we will carry out later, it is of crucial importance that, up to a common scaling factor $z \in \mathbb{Z}$, there is exactly one quadratic equation satisfied by a quadratic irrational u.

As this is very important when considering the resonances of the eigenvalues in Bianchi models, we include a proof of this fact here (and we assume that the c_i do not have a common factor because we will be interested in the smallest possible coefficients, where this is clear, see section 1.5):

LEMMA 0.1. *For a (fixed) quadratic irrational u, let $c_i \in \mathbb{Z}, i = 1...3$ be s.t. $c_1 + c_2 u + c_3 u^2 = 0$ and $\gcd(c_i) = 1$, i.e. the c_i do not have a common factor. Now assume that $d_1 + d_2 u + d_3 u^2 = 0$ also holds with $d_i \in \mathbb{Z}$. Then it follows that*

$$\exists z : d_i = z * c_i, \text{ for } (i = 1...3) \text{ with } z \in \mathbb{Z}$$

PROOF. Multiplying the equation with coefficients c_i with d_1 and the other one with c_1 results in the following two equations:

$$\begin{aligned} d_1 c_1 + d_1 c_2 u + d_1 c_3 u^2 &= 0 \\ c_1 d_1 + c_1 d_2 u + c_1 d_3 u^2 &= 0 \end{aligned}$$

Subtracting the second from the first equation leads to

(1) $$u(d_1c_2 - c_1d_2 + (d_1c_3 - c_1d_3)u) \;=\; 0$$

and, as $u \neq 0$, we conclude that

$$u = \frac{c_1d_2 - d_1c_2}{d_1c_3 - c_1d_3}$$

if $d_1c_3 - c_1d_3 \neq 0$, which leads to a contradiction because $u \notin \mathbb{Q}$ was assumed.

If, on the other hand, $d_1c_3 - c_1d_3 = 0$, it follows from (1) that also $d_1c_2 - c_1d_2 = 0$, which leads to the conclusion that $\frac{d_1}{c_1} = \frac{d_2}{c_2} = \frac{d_3}{c_3} := z$ with $z \in \mathbb{Z}$. Note that $z \in \mathbb{Q}$ would lead to a contradiction because we assumed that the c_i do not have a common factor. □

1.2. The Case of Bianchi IX. In order to check the (SNC) for the linearized vectorfield at a point on the Kasner circle, observe that $DX(p)$ is diagonal and that there are three hyperbolic eigenvalues for all points of the Kasner circle except for the Taub points.

In terms of the Kasner parameter u, the following formulas hold for those three eigenvalues (see section **??**):

(2) $$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{-6u}{1+u+u^2}, \frac{6(1+u)}{1+u+u^2}, \frac{6u(1+u)}{1+u+u^2} \right)$$

All 3 hyperbolic eigenvalues are real. A resonance thus means in this case: $\exists k = (k_1, k_2, k_3), k_i \in \mathbb{Z}$ s.t.

(3) $$k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0$$

where either all of the k_i must have the same sign, or the one of the k_i that has a different sign than the other two must be equal to ± 1. Because this "sign condition" will play an important role later on, let us make the following definition:

Definition. *A triple $k = (k_1, k_2, k_3), k_i \in \mathbb{Z}$ satisfies the Resonance Sign Condition (RSC) $\Longleftrightarrow$ either all of the k_i must have the same sign, or the one of the k_i that has a different sign than the other two must be equal to ± 1*

Only if a triple fulfils the RSC, it qualifies as a coefficient-triple for a resonance that prevents the application of Takens Linearization Theorem. This means that if we can show that resonant coefficients do not fulfill the RSC, they do not matter and Takens-Linearization is still possible. Note that a simple way of showing that the RSC is not satisfied is to show that one coefficient is strictly bigger than one, while a different one is strictly less than minus one.

1.3. SNC for Infinite Periodic Heteroclinic Chains. In preparation for further generalizations to Bianchi-models of class B, this section is formulated a bit more general that it would be necessary for discussing only the case of Bianchi IX. As seen above, the eigenvalues of the linearized vectorfield in BIX for points of the Kasner circle can be expressed in the Kasner parameter u:

$$\lambda_i = \frac{l_1^i + l_2^i u + l_3^i u^2}{1 + u + u^2} \tag{4}$$

Combining (3) and (4), one gets

$$k_1(l_1^1 + l_2^1 u + l_3^1 u^2) + k_2(l_1^2 + l_2^2 u + l_3^2 u^2) + k_3(l_1^3 + l_2^3 u + l_3^3 u^2) = 0 \tag{5}$$

or, equivalently,

$$(k_1 l_1^1 + k_2 l_1^2 + k_3 l_1^3) + (k_1 l_2^1 + k_2 l_2^2 + k_3 l_2^3)u + (k_1 l_3^1 + k_2 l_3^2 + k_3 l_3^3)u^2 = 0 \tag{6}$$

As discussed above, for infinite periodic heteroclinic chains, there are (up to a common scaling factor) unique coefficients $c_i \in \mathbb{Z}$ s.t.

$$c_1 + c_2 u + c_3 u^2 = 0 \tag{7}$$

Comparing (5) to (6), one sees that (SNC) does not hold if $\exists k = (k_1, k_2, k_3)$ as above and $z \in \mathbb{Z}$ s.t.

$$M * \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = z * \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{8}$$

with

$$M = \begin{pmatrix} l_1^1 & l_1^2 & l_1^3 \\ l_2^1 & l_2^2 & l_2^3 \\ l_3^1 & l_3^2 & l_3^3 \end{pmatrix}$$

where we will solve (8) for (k_1, k_2, k_3) in order to check the order of the first resonance.

1.4. Conclusions for Bianchi IX. It can be seen easily that the formulas (10) imply that for Bianchi IX we have

$$M_{BIX} = \begin{pmatrix} 0 & 6 & 0 \\ -6 & 6 & 6 \\ 0 & 0 & 6 \end{pmatrix} = 6 * \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$M_{BIX}^{-1} = \frac{1}{6} * \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe that we have a choice of the factor z on the right hand side of (8), and that a choice of $z = 6$ will result in an integer resonance with the smallest possible order:

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \frac{1}{6} * \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} * 6 * \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 + c_3 \\ c_1 \\ c_3 \end{pmatrix} \tag{9}$$

If the entries of the vector on the right hand side of (9) do not have a common factor, then the first resonance will occur at order $l := |k_1| + |k_2| + |k_3| = |c_1 - c_2 + c_3| + |c_1| + |c_3|$

1.5. Uniqueness of the Resonance. For the argument we will carry out later, it is of crucial importance that we find the order l of the **first** resonance, meaning that we can exclude all resonances with order $\tilde{l} < l$.

In order to do this, we will need the Lemma 0.1 on the uniqness of the coefficients for the quadratic equation for quadratic irrationals.

We claim that if we choose the smallest possible coefficients c_i for the equation in u (meaning that the c_i do not have a common factor), this will lead to the smallest resonance $l := |k_1| + |k_2| + |k_3|$.

This is true because of the linear dependence of the k_i on the c_i in (9), meaning that we can exclude all resonances with order $\tilde{l} < l$.

2. Continued Fraction Expansion for Quadratic Irrationals

We will use the following notation for continued fractions:

$$u = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}} =: [a_0, a_1, a_2, ...]$$

In this section, we will consider 3 classes of examples, namely $u \in \mathbb{R}$ with constant, 2-periodic and 3-periodic continued fraction expansions, i.e either $u = [a, a, ...]$ or $u = [a, b, a, b, ...]$ or $u = [a, b, c, a, b, c, ...]$ for $a, b, c \in \mathbb{N}$. We also recall from section **??** that the Kasner map has the following form:

$$u = \begin{cases} u - 1 & u \in [2, \infty] \\ \frac{1}{u-1} & u \in [1, 2] \end{cases}$$

2.1. Constant Continued fraction. Because of the form of the Kasner-map, starting with $u = [a, a, ...]$ will result in the following base-points on the Kasner-circle:

$$\begin{aligned} u_0 &= [a, a, a, ...] \\ u_1 &= [a-1, a, a, ...] \\ u_2 &= [a-2, a, a, ...] \\ &\dots \\ u_{a-1} &= [1, a, a, ...] \\ u_a &= [a, a, a, ...] \\ &\dots \end{aligned}$$

That's why we have to check the Non-Resonance-Conditions at all points with $u = [m, a, a, ...]$ for $m = 1...a$. Now note that for $u = [m, a, a, ...]$ it holds that

$$\frac{1}{u-m} - a = u - m$$

which means that

$$(m^2 - am - 1) + (a - 2m)u + u^2 = 0$$

resulting in a coefficient vector

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} m^2 - am - 1 \\ a - 2m \\ 1 \end{pmatrix}$$

Now we can use equation (9) to compute the coefficients for the resonance of the eigenvectors (we set $s = -1$ in order to match the condition (**??**)):

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = -1 * \begin{pmatrix} c_1 - c_2 + c_3 \\ c_1 \\ c_3 \end{pmatrix} = \begin{pmatrix} -m^2 + (a-2)m + a \\ -m^2 + am + 1 \\ -1 \end{pmatrix}$$

2.2. 2-Periodic Continued Fraction Expansion. For $u = [a, b, a, b, ...]$, we have to check the base-points with $u = [m, b, a, b, a, ...]$ with $m = 1...a$ and $u = [m, a, b, a, b, ...]$ for $m = 1...b$. Applying the same procedure as above, we note that that u satisfies

$$\frac{1}{\frac{1}{u-m} - a} - b = u - m \ \ \& \ \ \frac{1}{\frac{1}{u-m} - b} - a = u - m$$

when $u = [m, a, b, a, b, ...]$ and $u = [m, b, a, b, a, ...]$, respectively, and get the following coefficient vectors for u:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -am^2 + abm + b \\ 2am - ab \\ -a \end{pmatrix} \ \& \ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -bm^2 + abm + a \\ 2bm - ab \\ -b \end{pmatrix}$$

resulting in these coefficient vectors for the eigenvalues (we set $s = 1$ this time):

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -am^2 + (ab-2a)m + ab - a + b \\ -am^2 + abm + b \\ -a \end{pmatrix}$$

and

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -bm^2 + (ab-2b)m + ab + a - b \\ -bm^2 + abm + a \\ -b \end{pmatrix}$$

2.3. 3-Periodic Continued Fraction Expansion. In complete analogy to the computations above, we find the following formulas, for the 3 relevant cases. Note that we show the coefficient vectors for u below, and in all three cases we have to compute the coefficient vectors for the eigenvalues as done before:

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 + c_3 \\ c_1 \\ c_3 \end{pmatrix}$$

u=[m,b,c,a,...] for m=1...a.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} m^2 + mc + m^2bc - bm - am - ac - abcm - 1 \\ abc + a + b - c - 2m - 2mbc \\ 1 + bc \end{pmatrix}$$

u=[m,c,a,b,...] for m=1...b.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} m^2 + ma + m^2ca - cm - bm - ba - abcm - 1 \\ abc + b + c - a - 2m - 2mca \\ 1 + ca \end{pmatrix}$$

u=[m,a,b,c,...] for m=1...c.

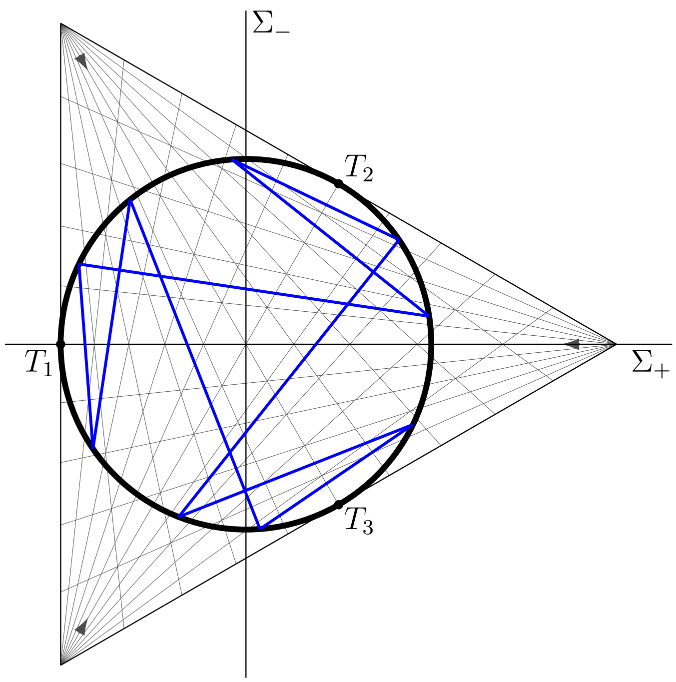
$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} m^2 + mb + m^2ab - am - cm - cb - abcm - 1 \\ abc + c + a - b - 2m - 2mab \\ 1 + ab \end{pmatrix}$$

3. Results on Admissibility of Periodic Heteroclinic Chains in Bianchi IX

In this section, we will concretely check the Sternberg Resonance Conditions for periodic heteroclinic chains in BIX.

We will prove some general theorems, while more concrete examples can be found in the Appendix 4.7.

3.1. Constant Continued Fraction Development. We will give a proof of the fact that there are no infinite periodic heteroclinic chains with constant continued fraction development that allow Takens - Linearization at their base points. More geometrically, this excludes "symmetric" heteroclinic chains with the same number of "bounces" near all of the 3 Taub Points - the result shows that we have to require some "asymmetry" in the bounces in order to allow for Takens-Linearization. Below, there is an illustration of the heteroclinic chain belonging to $u = [3, 3, ...]$ which does not allow for Takens-Linearization:



THEOREM 0.2. *For any heteroclinic chain with constant continued fraction development, Takens-Linearization fails at some base point.*

PROOF. As we have seen above, a periodic heteroclinic chain has a periodic continued fraction development, leading to a resonance, and let us call the coefficients for that resonance $k = (k_1, k_2, k_3)$. The first thing we have to check is if k satisfies the Resonance Sign Condition (RSC) defined above.

LEMMA. *For constant continued fraction development,* $(u = [a, a, ...])$, *the coefficient vector* $k = (k_1, k_2, k_3)$ *satisfies the Resonance Sign Condition (RSC) at all base points.*

PROOF. To prove the Lemma, we observe the following when looking at the formulas for constant continued fraction development in section 2.1:

- for $m = a$, it holds that $k = (1, a, -1)$
- for $m = a - 1$, $k = (-a, 1, -1)$
- for $1 \geq m < a - 1$ and $k = (k_1, k_2, k_3)$, it holds that $k_1, k_2 > 0$, while $k_3 = -1$

Thus, the RSC are satisfied in all cases, and the coefficient vector would qualify.

□

To prove Theorem 0.2, we have to compare two things:

- the order of the resonance of the eigenvalues at the basepoints, expressed first in the Kasner-parameter $(u = [a, a, ...])$ and then directly in a
- the required SNC for C^1-stable-manifolds, i.e. $\alpha(1)$ at all base points

The base points of a infinite periodic heteroclinic chain with $u = [a, a, ...]$ are $u = [m, a, ...]$ for $m = 1...a$. To prove the Theorem, it is enough to show the violation of the Sternberg Non-Resonance Conditions at one base point. Consider the case $m = a - 1$ and start with the formulas for the coefficient vectors, as computed above:

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -m^2 + (a-2)m + a \\ -m^2 + am + 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ a \\ -1 \end{pmatrix}$$

Therefor, it holds that $|k| = a + 2$, i.e. we have linear growth of $|k|$ in a.

On the other hand, re-consider the formulas for the eigenvalues in BIX:

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{-6u}{1 + u + u^2}, \frac{6(1+u)}{1 + u + u^2}, \frac{6u(1+u)}{1 + u + u^2} \right) \tag{10}$$

and order them according to magnitude (with the notation from the SNC's from the Takens-Theorem):

$$\begin{pmatrix} N \\ n \\ m = M \end{pmatrix} = \begin{pmatrix} |\lambda_1| \\ |\lambda_2| \\ |\lambda_3| \end{pmatrix}$$

Insert in the formulas for α, β and compute:

$$\beta = Ceiling[\frac{N + k(M+n)}{n}] \geq \frac{u^2 + 3u + 1}{u+1}$$

$$\alpha = Ceiling[\frac{M + \beta(N+m)}{m}] \geq \frac{u^3 + 5u^2 + 8u + 3}{u+1}$$

This shows quadratic growth for α in u. In fact, for $u = [a-1, a, ...] = a - 1 + \frac{1}{a+\frac{1}{a+\frac{1}{...}}}$, it holds $\forall a > 0 : |k| < \alpha(1)$, i.e. the SNCs are violated and Takens-Linearization is not possible, which proves the Theorem. For consistency, also compare to Appendix 4.7, where we used Mathematica to compute $\alpha(1)$ and $|k|$ for $u = [m, a, ...]$ for $m = 1...a$ and $a = 1...9$.

□

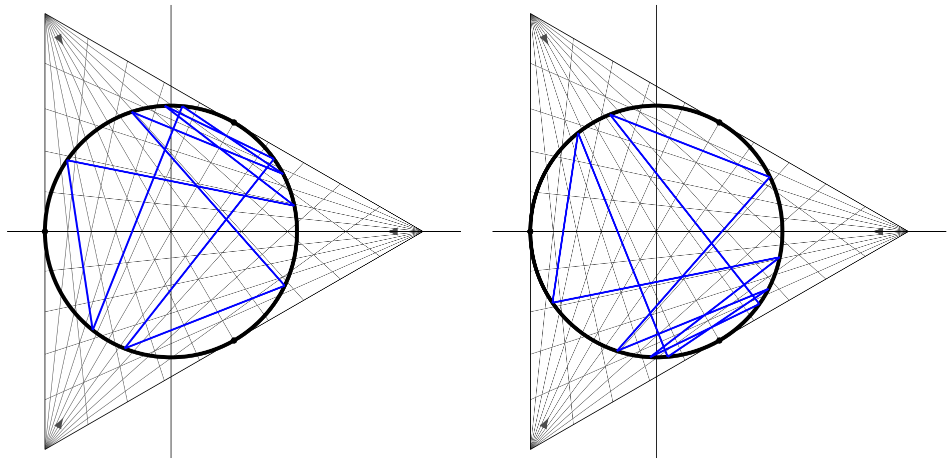
3.2. 2-Periodic Continued Fraction Development. In this section, we will prove the following Theorem:

THEOREM 0.3. *For admissible heteroclinic chains with 2-periodic continued fraction development, Takens Linearization is possible at all base points.*

Here, admissible means that the continued fraction developments has minimal period 2 and the entries are strictly bigger than one (even after cancelling out a possible common factor). To be precise, we define an admissible 2-periodic continued fraction development as follows:

DEFINITION 0.4. *A 2-periodic $u = [a, b, a, b, ...]$ is called admissible $\iff a, b > 1$ and neither $a \mid b$ nor $b \mid a$.*

Note that from the condition above, it follows in particular that $a \neq b$, beeing consistent with the results in the section above about constant contiuned fractions. Two examples of such a heteroclinic chains are illustrated below, with u=[3,2,3,2,...] and with u=[2,3,2,3,...], which are 10-cycles (also compare Appendix 1.2):



PROOF. The Theorem will directly follow from the following Lemma:

LEMMA 0.5. *For admissible 2-periodic continued fraction developments, the coefficient vector* $k = (k_1, k_2, k_3)$ *violates the Resonance Sign Condition (RSC) at all base points*

PROOF. When we look at the formulas for 2-periodic continued fraction development in section 2.2, we can observe the following:

- for $u = [m, a, b, a, b, ...]$ and $m = 1...b$, it holds that $k_3 = -a < -1$ and $k_2 \geq b > 1$ as $bm \geq m^2$
- for $u = [m, b, a, b, a, ...]$ and $m = 1...a$, it holds that $k_3 = -b < -1$ and $k_2 \geq a > 1$ as $am \geq m^2$

This means that the RSC are violated at all base points of the heteroclinic chain, and the lemma is proven. Note that we need $a, b > 1$, and that if we had $a \mid b$ or $b \mid a$, then coefficients k_1, k_2, k_3 would have a common factor we could cancel, leading to an earlier resonance. That's why we need to restrict to admissible 2-periodic continued fraction developments as defined above.

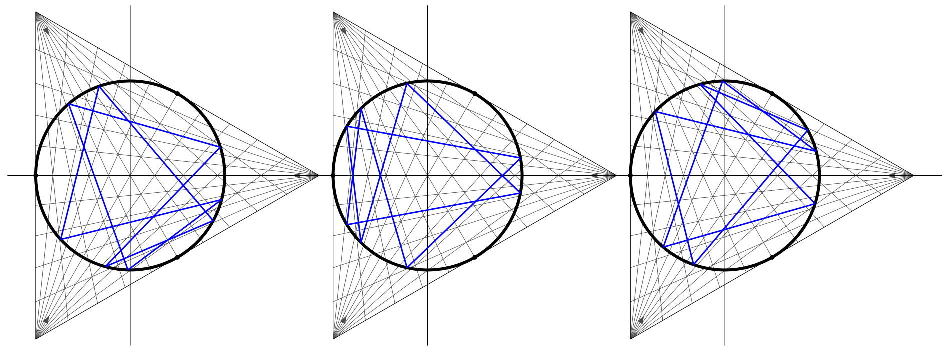
□

The Lemma shows that, for "sign reasons", the occuring resonaces are excluded and do not matter for the application of the Takens Theorem. Therefor Takens Linerarization is possible, as claimed in Theorem 0.3.

□

3.3. Continued Fraction Development with Higher Periods. The idea behind the proof of Lemma 0.5 can be generalized to continued fraction developments with higher periods. However, it is not so easy anymore to find conditions that assure in general that the resulting coefficients do not have a common factor. We will comment on this matter further at the end of the section.

At first, consider 3-periodic continued fractions. Three examples of such a heteroclinic chains are illustrated below, with u=[1,1,2,1,1,2,...], u=[1,2,1,1,2,1,...] and u=[2,1,1,2,1,1,...] which are 8-cycles and arguably the simplest examples of periodic heteroclinic chains where our method works (this can be checked directly for the concrete examples above, see Appendix 1.3). They all start in sector 5, and the different position of the number "2" in the contiued fraction development leads to bounces around the different Taub points which can be seen in the pictures below:



LEMMA 0.6. *Consider a continued fraction development with minimal period 3, i.e. with $u = [a, b, c, a, b, c, ...]$ and not $a = b = c$. Then the corresponding coefficient vector $k = (k_1, k_2, k_3)$ violates the Resonance Sign Condition (RSC) at all base points if the k_i do not have a common factor .*

PROOF. When we look at the formulas for 3-periodic continued fraction development in section 2.3, we can observe the following:

- for $u = [m, b, c, a, b, c, a, ...]$ and $m = 1...a$, it holds that $k_2 = c_1 \leq -bm - 1 < -1$ and $k_3 = c_3 = 1 + bc > 1$
- for $u = [m, c, a, b, c, a, b, ...]$ and $m = 1...b$, it holds that $k_2 = c_1 \leq -cm - 1 < -1$ and $k_3 = c_3 = 1 + ca > 1$
- for $u = [m, a, b, c, a, b, c, ...]$ and $m = 1...c$, it holds that $k_2 = c_1 \leq -am - 1 < -1$ and $k_3 = c_3 = 1 + ab > 1$

This means that the RSC are violated at all base points of the heteroclinic chain if we know that neither $k_2 \mid k_3$ nor $k_3 \mid k_2$. This is true in particular if the k_i do not have a common factor as we have assumed for convenience, so Lemma 0.6 is proven.

Note that if we had $a = b = c$, then coefficients k_1, k_2, k_3 would have a common factor, resulting in an earlier resonance as explained above. Also compare to Appendix 1.3 for a consistency check. $\square$

We now try to generalize the argument above to higher periodic continued fractions. In order to do this let us make some general definitions and observations (following **[37]** §19[1], compare also **[25]** §10):

For continued fractions of the form

$$u = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}} =: [a_0, a_1, a_2, ...]$$

we define the following numbers A_k, B_k recursively:

$$A_k = A_{k-1}a_k + A_{k-2}$$
$$B_k = B_{k-1}a_k + B_{k-2}$$

with $A_{-1} = 1, A_{-2} = 0$ and $B_{-1} = 0, B_{-2} = 1$, leading to $A_0 = a_1$, $A_1 = a_0a_1 + 1$ and $B_0 = 1$, $B_1 = a_1$.

For an (infinite) continued fraction, we define the "tails" as follows:

$$\xi_k := [a_k, a_{k+1}, ...]$$

Then we have the following general recursion formula for convergent infinite continued fractions $u = [a_0, a_1, ..., a_{k-1}, \xi_k]$ (and $k \geq 0$):

$$u = \xi_0 = \frac{A_{k-1}\xi_k + A_{k-2}}{B_{k-1}\xi_k + B_{k-2}}$$

, which can be proved by induction.[2]. Also compare **[25]** §2 and §3.

Now consider pre-periodic continued fractions with pre-period h and minimal period p, as made precise in the following definition:

DEFINITION. *We call u an h-pre-periodic continued fraction with pre-period h, minimal period p* $\iff u = [a_0, ...a_{h-1}, \overline{a_h, a_{h+1}, ..., a_{h+p-1}}]$ *with* $a_\nu = a_{\nu+p} \forall \nu \geq h$ *and* $\nexists \tilde{p} < p$ *s.t.* $a_\nu = a_{\nu+\tilde{p}} \forall \nu \geq h$

Note that it also holds that $\xi_\nu = \xi_{\nu+p} \forall \nu \geq h$. Thus we can get the following formulas (set $k = h$ and $k = h + p$):

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[1]but note we have a different labelling of the coefficients as we do not consider continued fractions with enumerators different from one

[2]For $k = 0$, the formula holds by definition: $\xi_0 = \frac{A_{-1}\xi_0 + A_{-2}}{B_{-1}\xi_0 + B_{-2}} = \frac{\xi_0}{1}$.

$$\xi_0 = \frac{A_{h-1}\xi_h + A_{h-2}}{B_{h-1}\xi_h + B_{h-2}}$$

and

$$\xi_0 = \frac{A_{h+p-1}\xi_{h+p} + A_{h+p-2}}{B_{h+p-1}\xi_{h+p} + B_{h+p-2}} = \frac{A_{h+p-1}\xi_h + A_{h+p-2}}{B_{h+p-1}\xi_h + B_{k+p-2}}$$

By solving both equations for ξ_h, we get the following quadratic equation for ξ_0 :

$$c_3\xi_0^2 + c_2\xi_0 + c_1 = 0$$

with (we abbreviate $g = h + p$)

$$\begin{aligned} c_3 &= B_{h-2}B_{g-1} - B_{h-1}B_{g-2} \\ c_2 &= B_{h-1}A_{g-2} + A_{h-1}B_{g-2} - A_{h-2}B_{g-1} - B_{h-2}A_{g-1} \\ c_1 &= A_{h-2}A_{g-1} - A_{h-1}A_{g-2} \end{aligned}$$

The formulas above specialize to (for $h = 0$, this corresponds to the formula for periodic continued fractions without pre-period)

$$\begin{aligned} c_3 &= B_{p-1} \\ c_2 &= B_{p-2} - A_{p-1} \\ c_1 &= -A_{p-2} \end{aligned}$$

and for $h = 1$ to[3]

$$\begin{aligned} c_3 &= -B_{p-1} \\ c_2 &= A_{p-1} + a_0B_{p-1} - B_p \\ c_1 &= A_p - a_0A_{p-1} \end{aligned}$$

Now we are in a position to state the main aim of this section:

Conjecture 0.7. *Let $u = [a_0, a_1, ...]$ be an (infinite) periodic continued fraction with minimal period $p \geq 3$. Then the corresponding heteroclinic chain allows Takens-Linearization at all base points.*

Proof. (idea of proof, but note the remark below)
Let $u = [a_0, a_1, ...]$ be an (infinite) periodic continued fraction. We need to show that the NRC's are satisfied at all base points of the heteroclinic chain. Because of the form of the Kasner-map, we have to check all

[3]compare to the formulas for p=1,2,3 presented in section 2.3, as a consistency check

Kasner-parameters of the form $u = u_m = [m, \overline{a_1, a_2, ..., a_{p-1}, a_p}]$, i.e. it holds that $a_0 = m$ (with $1 \leq m \leq a_p$) and $a_\nu = a_{\nu+p}$, but now only $\forall \nu \geq 1$. From the formulas above (case $h = 1$) we observe the following for the corresponding coefficients of the resonances of the eigenvalues:

$$k_3 = c_3 = -B_{p-1} < -1$$

where we need our assumption that $p \geq 3$ as $B_1 = a_1$ which might be one, but $B_2 = a_2 a_1 + 1$ which is bigger than one. Also

$$k_2 = c_1 = A_p - a_0 A_{p-1} = (a_p - a_0) A_{p-1} + A_{p-2} > 1$$

because we know that $a_0 = m \leq a_p$ and $A_1 = a_0 a_1 + 1$ is bigger than one. That's why the "Resonance Sign Condition" is violated at all base points, and Takens-Linearization is possible. □

The reason why we don't call the Conjecture above a Theorem is that we are not able to exclude in general that c_1 divides c_3 or vice versa, which is essential for the proof above to work out. We believe it is possible to prove this in general for most periodic continued fraction with minimal period $p \geq 3$, probably with a small set of exceptions, but this is an issue for further research.

4. Details on the Proof for Stable Manifolds

In this section, we complete the proof of Theorem **??** by showing that there is a C^1-hyperbolic structure for the return map in Bianchi IX after linearizing at all base points of a heteroclinic chain. This then leads to a C^1-stable manifold, as claimed.

We procced along the lines and very close to the paper of Béguin [**2**], but we adapt the notation to our needs and the situation of a periodic chain that Béguin does not consider.

Also compare to the papers by Liebscher et al. [**27, 28**], where they work in a Lipschitz-setting without linearizing at the Kasner circle. There, the following return maps are considered

$$\Phi_k^{return} = \Phi_k^{glob} \circ \Phi_k^{loc} : \Sigma_k^{in} \to \Sigma_{k+1}^{in}$$

where the index k stands for the base points on the Kasner circle of the heteroclinic chain, i.e. Φ_k^{return} maps from one In-section to the next. It is shown that those maps satisfy the necessary cone conditions to allow for a graph-transform on Lipschitz-graphs on a subset of Σ^{in} including the origin (which stands for the heteroclinic orbit). This then leads to the stable manifold result.

However, like Béguin [**2**], we will use a collection Φ_B^{return} of these return maps for all base points of the set $B \subset \mathcal{K}$. We then show that there exists a C^1- hyperbolic structure for a suitable subset of the corresponding In-sections Σ_B^{in}. This results in a C^1-stable manifold.

4.1. Application of Takens Theorem. Let $B = \{p_1, ..., p_n\}$ be the collection of base points on the Kasner circle of the periodic heteroclinic chain we are looking at. Then, as we have chosen an admissible periodic chain that satisfies the necessary Non-Resonance-Conditions by assumption, we can chose co-ordinates near each point $p_k \in B$ such that the vector field has the form described by the Takens Theorem, i.e. it is essentially linear in a neighbourhood U_{p_k}. More precisely, the application of Takens Linearization Theorem is done in the following form (compare Béguin, p.10):

THEOREM 0.8. *Let $p \in B$ be any point of the set of admissible base points B. Then there exists a Takens-Neighbourhood U_p of p in the phase-space of the Wainwright-Hsu ODEs $\mathcal{W}$ and a C^1-coordinate-system on U_p such that the Wainwright-Hsu vector field X^W can be written as*

$$X^W(x^c, x^s, x^{ss}, x^u) = \lambda_s(x^c)x^s \frac{\partial}{\partial x^s} + \lambda_{ss}(x^c)x^{ss} \frac{\partial}{\partial x^{ss}} + \lambda_u(x^c)x^u \frac{\partial}{\partial x^u}$$

where $\lambda_{ss}(x^c) < \lambda_s(x^c) < 0 < \lambda_u(x^c)$ *for all* x^c.

PROOF. A direct application of the Takens-Theorem **??** (from chapter **??**) gives the existence of a coordinate system $(x^c, x^{s1}, x^{s2}, x^u)$ in U_p s.t. X^W has the following form in these coordinates:

$$X^W(x^c, x^{s1}, x^{s2}, x^u) = \phi(x^c)\frac{\partial}{\partial x^c} + \sum_{i,j=1}^{2} a_{ij}(x^c)y^{si}\frac{\partial}{\partial y^{sj}} + b(x^c)x^u\frac{\partial}{\partial x^u}$$

For the vector field X^W in the original coordinates, the set $\mathcal{K}\cap U_p$ is the local center-manifold in the neighbourhood U_p at the point p, and it consists of equilibria. As the vector field above vanishes on $K = \{x^{s1} = x^{s2} = x^u = 0\}$ and nowhere else, it follows that $K = \mathcal{K} \cap U_p$. This also means that $\phi \equiv 0$ in the neighbourhood U_p, i.e. there is no drift at all in the center-direction. Now fix $\{x^c = \xi\}$. As can be seen from the formula above, the vector field $X^W(x^c, x^{s1}, x^{s2}, z^u)$ is linear on the restruction to this submanifold. A linear change of coordinates then diagonalizes the 2×2-matrix $(a_{ij})_{i,j\in\{1,2\}}$, as we have 2 distinct real stable eigenvalues of X^W at the point $(\xi, 0, 0, 0)$, and this diagonalization can be done simultaneously, as eigenvalues and eigendirections depend in a smooth way on ξ. Label these new coordinates (x^c, x^s, x^{ss}, x^u) and observe that we have found the claimed local form of the vector field

$$X^W(x^c, x^s, x^{ss}, x^u) = \lambda_s(x^c)x^s\frac{\partial}{\partial x^s} + \lambda_{ss}(x^c)x^{ss}\frac{\partial}{\partial x^{ss}} + \lambda_u(x^c)x^u\frac{\partial}{\partial x^u}$$

□

For the rest of the section, we will use the follwing coordinates: Near the Kasner-circle, we take the coordinates given by the Takens-Linearization-Theorem, at each base point p_k of the heteroclinic chain, and otherwise, we stick to the coordinates $N_i, \Sigma_{+/-}$ of the Wainwright-Hsu-System. The different coordinate systems give rise to the following metrics: the Riemanian metric $g_p = dx^c \wedge dx^c + dx^s \wedge dx^s + dx^{ss} \wedge dx^{ss} + dx^u \wedge dx^u = (dx^c)^2 + (dx^s)^2 + (dx^{ss})^2 + (dx^u)^2$ for the Takes-coordinates in a neighbourhood U_p near a point p of the Kasner circle, and the Riemanian metric $h = dN_1^2 + dN_2^2 + dN_3^2 + d\Sigma_+^2 + d\Sigma_-^2$. Later we use a "global" Riemannian metric adapted to our set of base points B by defining g_B such that

$$g_B \restriction U_p = g_p \forall p \in B \tag{11}$$

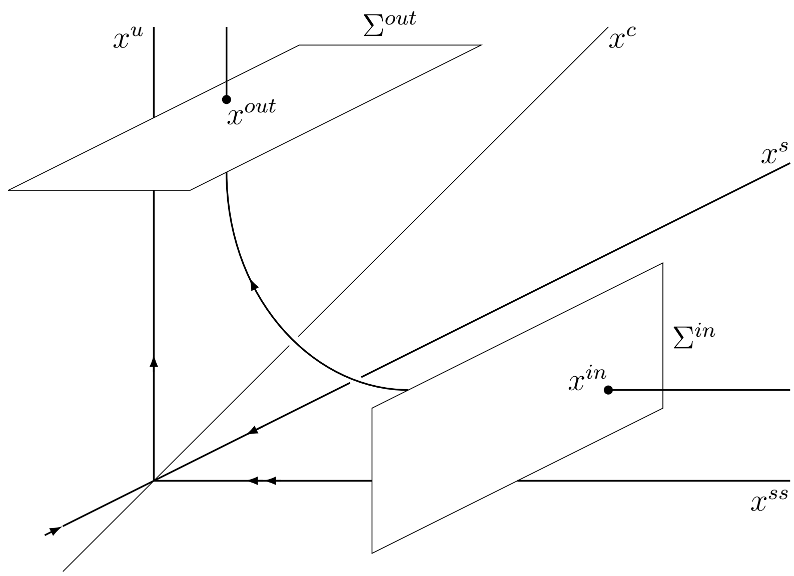

FIGURE 1. Local passage Φ^{loc}.

For the local passage, which we will consider next, we are entirely in the neighbourhood U_p and can use the "local" metric $g_p = (dx^c)^2 + (dx^s)^2 + (dx^{ss})^2 + (dx^u)^2$.

4.2. Local Passage. Our next step is to deal with the local passage near an equilibrium of the Kasner circle $\mathcal{K}$. Figure 1 shows a graphic illustration of the situation in Bianchi IX - note that we are in the lucky situation here that the incoming stable eigenvalue is always stronger that the outgoing unstable eigenvalue, this will change in $BVI_{-\frac{1}{9}}$ that we deal with in chapter **??**.

Now we come to the definition of the local In- and Out-Sections illustrated in the picture above: For a point $p_k \in B$, we first define the box V_p as $V_p = V_p(\alpha, \beta, \epsilon) := \{q = (x^c_q, x^s_q, x^{ss}_q, x^u_q) \in U_p | 0 \leq x^s_q, x^{ss}_q, x^u_q \leq \epsilon, \alpha \leq x^c_q \leq \beta\}$ and α, β, ϵ are chosen so small the box V_p lies completely inside the Takens-neighbourhood U_p. Denote their union by $V_B = \bigcup_{k=1}^n V_{p_k}$.

Then define the sections by $\Sigma_k^{in,ss} := V_{p_k} \cap \{x^{ss} = \epsilon\}$ and $\Sigma_k^{out} := V_{p_k} \cap \{x^u = \epsilon\}$. Finally define the "collections" of sections for the whole set of base-points B: $\Sigma_B^{in,s} = \bigcup_{k=1}^n \Sigma_k^{in,s}$,$\Sigma_B^{in,ss} = \bigcup_{k=1}^n \Sigma_k^{in,ss}$ and $\Sigma_B^{out} = \bigcup_{k=1}^n \Sigma_k^{out}$, and finally $\Sigma_B^{in} = \Sigma_B^{in,s} \cup \Sigma_B^{in,ss}$.

We need some more notation before we can introduce the main theorem of this section. Decompose the tangent spaces of the sections defined above into the parts of the hyperbolic direction (V^h, W^h), on the one hand, and the center-component (V^c,W^c), on the other hand. For this, let $q \in \Sigma_B^{in,ss}$ and $r \in \Sigma_B^{out}$:

- $T_q\Sigma_B^{in,ss} = V_q^h \oplus V_q^c$, where $V_q^h = span\{\frac{\partial}{\partial x^s}(q), \frac{\partial}{\partial x^u}(q)\}$, i.e. the additional stable direction and the unstable direction, and $V_q^c = span\{\frac{\partial}{\partial x^c}(q)\}$
- $T_r\Sigma_B^{out} = W_r^h \oplus W_r^c$, where $W_r^h = span\{\frac{\partial}{\partial x^{ss}}(r), \frac{\partial}{\partial x^s}(r)\}$, i.e. the both stable directions, because we are in the out-section, and $W_r^c = span\{\frac{\partial}{\partial x^c}(r)\}$

Note that one point of this construction is to "collect" also the tangent spaces like the other objects before, i.e. to talk about the decomposition of the tangent bundle of the set Σ_B^{out}, which is possible because all object depend smoothly on the base point:

- $T\Sigma_B^{out} = V^h \oplus V^c$
- $T\Sigma_B^{out} = W^h \oplus W^c$

Now we are in the position to state the theorem about the local passage. Recall that H_B stands for the set of all heterclinic Bianchi-II-orbits connecting base points of the set B (see chapter **??**, section **??**):

THEOREM 0.9. *Assume that, for all $p \in B$, the vector field has been according brought to the form as in the conclusion of Theorem 0.8. The local passage map $\Phi_B^{loc} : \Sigma_B^{in} \to \Sigma_B^{out}$ is a C^1-map that satifies, for $q \in H_B \cap \Sigma_B^{in}$:*

- *Φ_B^{loc} contracts super-linearly in the hyperbolic directions, i.e. $d\Phi_B^{loc}(q)(v) = 0 \; \forall v \in V_q^h$*
- *Φ_B^{loc} is the identity in the center-direction, i.e.*
  1. $d\Phi_B^{loc}(q)(V_q^c) = W^c_{\Phi_B^{glob}(q)}$
  2. $||d\Phi_B^{loc}(q)(v)||_{g_p} = ||v||_{g_p} \forall v \in V_q^c$

PROOF. Let $p \in B$ be a point from the set of admissible base points. Because of Theorem 0.8, the local passage near the Kasner circle Φ_p^{loc} in a neighbourhood U_p can be calculated explicitly (with $x_{in}^{ss} = 1$ in Σ_p^{in} and $x_{out}^u = 1$ in Σ_p^{out} after approriate scaling):

$$x_{out}^s = e^{\lambda_s t_{loc}} \cdot x_{in}^s = (x_{in}^u)^{-\frac{\lambda_s}{\lambda_u}} \cdot x_{in}^s \tag{12}$$

$$x_{out}^{ss} = e^{\lambda_{ss} t_{loc}} \cdot x_{in}^{ss} = (x_{in}^u)^{-\frac{\lambda_{ss}}{\lambda_u}} \tag{13}$$

$$x_{in}^u = e^{-\lambda_u t_{loc}} \cdot x_{out}^u \tag{14}$$

By solving the third equation for the local passage time t_{loc}, one obtains the following formulas for $\Phi_p^{loc} : \Sigma_p^{in} \to \Sigma_p^{out}$ (when $x^u > 0$):

$$\Phi_p^{loc}(x^c, x^s, 1, x^u) = (x^c, (x_{in}^u)^{-\frac{\lambda_s}{\lambda_u}} \cdot x_{in}^s, (x_{in}^u)^{-\frac{\lambda_{ss}}{\lambda_u}}, 1)$$

and for $x^u = 0$, we get (when following the heteroclinic orbit)

$$\Phi_p^{loc}(x^c, x^s, 1, 0) = (x^c, 0, 0, 1)$$

As the above equations show, the main point for understanding the local passage is the relation of the eigenvalues. In Bianchi IX, we know that it holds (away from the Taub points):

$$|\lambda_u| < |\lambda_s| < |\lambda_{ss}|$$

, i.e. the absolute value of the unstable eigenvalue is strictly smaller than the absolute value of the two stable eigenvalues. This can be seen from the formulas (10) expressing the eigenvalues in terms of the Kasner parameter u, see chapter **??**, section **??**). That's why it holds for the fractions which appear in the exponents of the formulas above:

$$-\frac{\lambda_s}{\lambda_u}, -\frac{\lambda_{ss}}{\lambda_u} > 1$$

and observe that both are necessarily positive because stable and unstable eigenvalues have opposite signs (note that this is even indepent of the chosen time direction towards/away from the big bang). This yields the claimed C^1-map and the super-linear contraction in the hyperbolic directions for the map Φ_p^{loc}.

As the vector field is completely linear in the Takens-neighbourhood, it trivially holds that $x^c_{out} = e^0 \cdot x^c_{in}$, i.e. we have not drift and Φ^{loc} is just the identity in the center-direction.

These observations hold for the local passage $\Phi_p^{loc} : \Sigma_p^{in} \to \Sigma_p^{out}$ at any admissible base point $p \in B$, and therfor also for the collection $\Phi_B^{loc} : \Sigma_B^{in} \to \Sigma_B^{out}$. □

4.3. Global Passage. Now we deal with the global passage. For the proof of the main theorem in this section, we consider two maps which map from the respective sections onto the Kasner circle by following the heteroclinic orbit (compare [**2**], p.19):

$\alpha : H_B \cap \Sigma_B^{out} :\to \mathcal{K} \cap V_B$

$\omega : H_B \cap \Sigma_B^{in} :\to \mathcal{K} \cap V_B$

where we recall that H_B stands for the Bianchi-II-heteroclinics and V_B is the collection of Takens-neighbourhoods (or the boxes, more precisely) constructed above when dealing with the local passage. At this point, we recall how we defined our global metric g_B, see (11). It is composed of the Riemanian metric $g_p = (dx^c)^2 + (dx^s)^2 + (dx^{ss})^2 + (dx^u)^2$ for the Takes-coordinates in a neighbourhood U_p near a point $p \in B$ of the Kasner circle, and the Riemanian metric $h = dN_1^2 + dN_2^2 +$

$dN_3^2 + d\Sigma_+^2 + d\Sigma_-^2$ otherwise. We may assume that both metics coincide when restricted to $\mathcal{K} \cap U_{p_i}$, because the local vector field has no center-component at the Kasner circle, i.e. one can replace center coordinate x by $\phi(x)$ for a diffeo ϕ without changing the vector field.

This means that both maps α, ω are local C^1-isometries for the metrics induced by the global metric g_B on the sets above, and we will use this fact in our proof below.

THEOREM 0.10. *There exits a neighbourhood $\mathcal{V}$ of $H_B \cap \Sigma_B^{out}$ in Σ_B^{out} such that the global passage map*

$$\begin{array}{rcl} \Phi_B^{glob} : \Sigma_B^{out} & \to & \Sigma_B^{in} \\ \mathcal{V} & \to & \Phi_B^{glob}(\mathcal{V}) \end{array}$$

is a C^1-map on $\mathcal{V}$ and a diffeomorphism onto its image.

Φ_B^{glob}*expands in the center direction, i.e. for $r \in H_B \cap \Sigma_B^{out}$, it satisfies*

(1) $d\Phi_B^{glob}(r)(W_q^c) = V^c_{\Phi_B^{glob}(r)}$

(2) $\exists \kappa > 1 : ||d\Phi_B^{glob}(r)(w)||_{g_B} \geq \kappa ||w||_{g_B} \forall w \in W_r^c$

PROOF. We know that for Ordinary Differential Equation with differentiable (C^k-)vector field, there is a differentiable (C^k-)dependence of the solution on the initial conditions (see e.g. [**1**]). This means that in general, for any "time-t-map" of a differentiable flow, for fixed $t = t^*$ and an open subset $U \subset \mathbb{R}^n$ of the phase space, we get a diffeomorphism onto its image:

$$\begin{array}{rcl} \phi_{t^*} : \mathbb{R}^n & \to & \mathbb{R}^n \\ U & \to & \phi_{t^*}(U) \end{array}$$

The Wainwright-Hsu vector field X^W is polynomial, hence analytic, that's why its flow $\phi_t(x_0)$ does depend in a differential (and even analytic) way on the inital condition. This means that the map $\Phi_p^{glob} : \Sigma_p^{out} \to \Sigma_{f(p)}^{in}$ is a C^1-map and a diffeomorphism onto its image, as claimed for the hyperbolic directions. We are left to show the second part of the theorem, dealing with the center directions. Now let $q \in H_B \cap \Sigma_B^{out}$. Then we observe that $\omega(\Phi^{glob}(q) = \omega(q) = f(\alpha(q)$, where f stands for the Kasner map. Because we have shown that both α and ω are local C^1-isometries w.r.t. g_B, we are left to prove that

$$\exists \kappa > 1 : \forall p \in B, \forall v \in T_p\mathcal{K} : ||df(p)(v)||_g \geq \kappa \cdot |v|_g$$

, which follows directly from the definition of the Kasner map, as we consider a periodic chain which clearly keeps a minimal distance from the Taub points, where f is not expanding.

These observations hold for the global passage $\Phi_p^{glob} : \Sigma_p^{out} \to \Sigma_{f(p)}^{in}$ at any admissible base point $p \in B$, and therfor also for the collection $\Phi_B^{loc} : \Sigma_B^{in} \to \Sigma_B^{out}$. □

4.4. The Return Map and the Hyperbolic Structure. As a consequence, we get the following result:

THEOREM 0.11. *The return map* $\Phi_B^{return} = \Phi_B^{glob} \circ \Phi_B^{loc} : \Sigma_B^{in} \to \Sigma_B^{in}$ *is a* C^1*-map that satisfies, for* $q \in H_B \cap \Sigma_B^{in}$

- Φ_B^{return} contracts super-linearly in the hyperbolic directions, i.e. $d\Phi_B^{return}(q)(v) = 0 \forall v \in V_q^h$
- Φ_B^{return}expands in the center direction, i.e. $\exists \kappa > 1 :$ $||d\Phi_B^{return}(q)(v)||_{g_B} \geq \kappa ||v||_{g_B} \forall v \in V_q^c$

PROOF. We recall the main idea behind our construction: We have shown that for the hyperbolic directions, the local passage is a contraction, while the global passsage is a diffeomorphism. Because of the differential dependence of a solution of an ODE on the initial conditions, the passage time for global passage near a heteroclinic orbit depends in a C^1-way on the base point considered. When approaching the attactor, it remains bounded, while the passage time for the local passage tends to infinity. That's why the local passage dominates, and we get a contraction in the hyperbolic directions. In the center direction, the local passage is the identity in our local coordinate system, which yields the claimed expansion when combined with the global passage which expands the center direction. More formally, we use the chain rule $d\Phi_B^{return}(v) = d\Phi_B^{glob}(\Phi_B^{loc}) \circ d\Phi_B^{loc}(v)$ to get the claims directly from our theorems above, for $q \in H_B \cap \Sigma_B^{in}$:

$d\Phi_B^{return}(q)(v) = 0 \; \forall v \in V_q^h$

$||d\Phi_B^{return}(q)(v)||_{g_B} \geq \kappa ||v||_{g_B} \forall v \in V_q^c$ □

The theorem above means that our return map Φ_B^{return} has a C^1-hyperbolic structure on the set $H_B \cap \Sigma_B^{in}$, i.e. that it is a hyperbolic set. Via Theorem 0.13 (described below), this C^1-hyperbolic structure leads to a C^1-stable-manifold.

To make this more precise, consider a point $p \in B$ and observe that the heteroclinic orbit $H_{p,f(p)}$ intersects Σ_B^{in} in exactly one point that we denote by q. We also note that $q \in (H_B \cap \Sigma_B^{in})$, i.e. it belongs to our hyperbolic set. Theorem 0.13 yields a C^1-embedded 2-dimensional stable manifold $W_\epsilon^s(\Phi, q)$ in Σ_B^{in}. And as we know that the orbits of the Bianchi IX flow are transversal to Σ_B^{in}, we obtain a 3-dimensional stable manifold for the base point p on the Kasner circle as claimed (compare [**2**], p. 22).

In summary, we arrive at the following theorem, which is equivalent to Theorem **??**:

THEOREM 0.12. *(Stable Manifolds for Points in B)*
Let $p \in B$, where B is the set of base points of a periodic heteroclinic chain that satisfies the Sternberg Non-Resonsonance-Conditions. Then there exists a three dimensional C^1-stable manifold $W^s(p)$ of initial conditions such that the corresponding vacuum Bianchi IX - solutions converge to the periodic heteroclinic chain towards the big bang.

Combining this with Theorem 0.3 and Definition 0.4 on the admissibility of 2-periodic continued fraction developments leads immediately to Theorem **??**.

Untill now, we have only dealt with periodic heteroclinic chains, as this was the "missing case" in the paper by Béguin, who was treating aperiodic chains. When we combine the two results, we can get C^1-stable manifolds for any points $p \in \mathcal{K}$ that do not contain "forbidden" base points in the closure of the orbit of p under the Kasner map f, i.e. $\overline{\{f^n(p)\}} \subseteq B_\epsilon^T$. For this we define B_ϵ^T to be the set of base points that satisfies the Non-Resonsonance-Conditions in order to allow for Takens Lineraization and keeps a minimum distance of ϵ from the Taub points. This second condition is trivially fullfilled for periodic chains and necessary in order to achive uniform rates of expansion/contraction for the hyperbolic structure. The reason is that both the expansion of the Kasner map as well as the contraction of the local passage breaks down at the Taub points.

We can also elaborate a bit about what it means that solutions of Bianchi IX converges to a heteroclinic chain towards the big bang. For example, we can show that the Hausdorff distance between the heteroclinic orbits that are part of the chain and the respective piece of the Bianchi IX-orbit tends to zero. This follows from the continuity of the flow and the properties of the stable manifold (see [**2**], p.21). Thus the limit of the analysis presented here can be formulated as in Theorem **??**.

4.5. C^1-Stable Manifolds for C^1-Hyperbolic Sets. We have shown that the global return map admits a C^1-hyperbolic structure. Béguin then uses the following Theorem (see [**2**], p.18) to prove the existence of a C^1-stable manifold: Theorem 0.13 shows that a C^1-Hyperbolic Structure leads to a C^1- stable manifold, where the "index s" of the hyperbolic set stands for the dimension of the stable subbundle of the tangen bundle TM (i.e. $s = dim(X_p)$ in the notation of Definition **??**). In addition, the theorem specifies the dependence of this manifold on the base point as well as the convergence rate:

THEOREM 0.13. *Let $\Phi : M \to M$ be a C^1map on a manifold M, and C be a compact subset of M which is a hyperbolic set of index s for the map Φ. Then, for every ϵ small enough, for every $q \in C$, the set*

$$W^s_\epsilon(\Phi, q) := \{r \in M | dist(\Phi^n(r), \Phi^n(q)) \leq \epsilon \, for \, every \, n \geq 0\}$$

is a C^1embedded s-dimensional disc, tangent to F^s_q at q, depending continuously on q (for the C^1topology on the space of embeddings). Moreover, if μ is a contraction rate for Φ on C, then there exists a constant κ such that, for every ϵ small enough, for every $q \in C$ and every $r \in W^s_\epsilon(\Phi, q)$

$$dist_g(\Phi^n(r), \Phi^n(q)) \leq \kappa\mu^n$$

Béguin names the book [**36**] by Palis and Takens (page 167) as a reference for Theorem 0.13. In this section of the Appendix "Hyperbolicity: Stable Manifolds and Foliations", the authors deal with hyperbolic sets for endomorphisms, but results are only sketched and no proofs included. However, there are classic sources for stable manifold theorems of hyperbolic sets: Partly based on an earlier paper ([**22**]), Hirsch and Pugh prove such a theorem in [**23**], which is a chapter of the book "Global Analysis" collecting the proceedings a symposium held on the topic in Berkeley, California, in 1968, and seems to be the first time such a result is proved. We will introduce the theorem by Hirsch/Pugh below, it can be used instead of 0.13 in order to prove our Theorem 0.12.

4.6. Generalized Stable Manifold Theorem by Hirsch/Pugh.

THEOREM. *(Generalized Stable Manifold Theorem) Let U be an open set in a smooth manifold $M(dim < \infty)$ and $f : U \to M$ a C^1-map. Let $\Lambda \subset U$ be a compact hyperbolic set and call the invariant splitting $T_\Lambda M = E_1 \oplus E_2$. Then there is a neighbourhood V of Λ, and* *submanifolds* $W^s(x), W^u(x)$ *tangent to* $E_2(x)$ *and* $E_1(x)$ *respectively for each* $x \in \Lambda$ *such that*

$$W^s(x) = \{y \in V | \lim_{n \to \infty} d(f \restriction V)^n y, f \restriction V)^n x) = 0\}$$

If f *is* C^k*, so is* $W^s(x)$*, and it depends continously on* f *in the* C^k*-topology. Moreover,* $W^s(x)$ *and its derivatives along* $W^s(x)$ *up to order* k *depend continously on* x*. In addition, there exist numbers* $K > 0, \lambda < 1$ such that if $x \in \Lambda, z \in W_x$ and $n \in \mathbb{Z}_+$ then *the following holds:*

$$d(f^n(x), f^n(z) \leq K\lambda^n$$

In **[23]**, the proof of the generalized stable manifold theorem is outlined as follows:

(1) Let $E = E_1 \times E_2$ be a Banach space; $T : E \to E$ a hyperbolic linear map expanding along E_1 and contracting along E_2; $E(r) \subset E$ the ball of radius r, and $f : E(r) \to E$ a Lipschitz pertubation of $T \restriction E(r)$. The unstable manifold W for f will be the graph of a map $g : E_1(r) \to E_2(r)$ which satisfies $W = f(W) \cap E(r)$. Then the following map Γ_f is considered (in a suitable function space G of maps g):

$$graph[\Gamma_f(g)] = E(r) \cap f(graph[g])$$

i.e. Γ_f is the graph transform of g by f. The fixed point g_0 of Γ_f gives the unstable manifold of f - its existence is proved by the contracting map principle if f is sufficiently close to T pointwise, and the Lipschitz constant of $f - T$ is small enough.

(2) If f is C^k so is g_0, which is proved by induction on k. The successive approximations $\Gamma_f^n(g)$ converge C^k to g_0 - here the Fibre Contraction Theorem is used.

(3) Let $\Gamma \subset U$ be a hyperbolic set. Let $\mathcal{M}$ be the Banach manifold of bounded maps $\Lambda \to M$, and $i \in \mathcal{M}$ the inclusion of Λ. Let $\mathcal{U} = \{h \in \mathcal{M} | h(\Lambda) \subset U\}$. Define $f_* : \mathcal{U} \to \mathcal{M}$ by

$$f_*(h) = f \circ h \circ f^{-1}$$

Then f_*has a hyperbolic fixed point at i. By the first point, f_* hast a stable manifold $\mathcal{W}^s \subset \mathcal{M}$. For each $x \in \mathcal{M}$, define $W^s(x) = ev_x(\mathcal{W}^s) = \{y \in M | y = \gamma(x)$ for some $\gamma \in \mathcal{W}^s\}$. This yields a system of stable manifolds for f along Λ

Point (1) of the outline above involves a graph-transform of Lipschitz-graphs (see e.g. [**44**], and compare also [**27, 28**], where it is described in detail how a graph transform can be used to prove Lipschitz-stable-manifolds in Bianchi models even without linearizing at the Kasner circle). Point (3) reduces the proof of a stable manifold for a hyperbolic set to the case of a fixed point, in a suitable chosen infinite-dimensional space (compare also [**36**], p.157).

4.7. Differentiability of the Stable Manifold. In step (2) above, the differentiability of the stable manifold is proved by the Fibre Contraction Principle (see [**23**], p.136 or [**24**], p.25). As the differentiability of the stable manifold is the main point of our Theorem **??**, we will comment a bit how this is done. For the invariant section (which will be the desired stable manifold) to be differentiable, it is not enough to obtain a fibre contraction. One important point is that it may not contract more along the base space than along the fibres (compare [**24**], p.26), otherwise there are examples where there is no differentiable invariant section (see e.g. [**44**], p. 435). That's why we need additional conditions that assure that the contraction on fibres is stronger than the contraction in the base space to prove a "C^r section Theorem" ([**44**], p. 436).

An alternative approach is the method of cones (e.g. taken by Robinson [**44**], p.185). As above, a stable manifold that is only Lipschitz is obtained in a first step, and then it is shown that the obtained manifold is infact C^k if the original map has this smoothness property ([**44**], p.194).

Finally, the book [**49**] also contains stable manifold theorems both for fixed points (chapter 5) and hyperbolic sets (chapter 6), in an abstract setting similar to [**23**], and also deals with the differentiability question (see [**49**], p.39).

Summary

We have shown that there are periodic heteroclinic chains in Bianchi IX for which there exisist C^1- Stable - Manifolds of orbits that follow these chains towards the big bang. This result is new, and should be compared with the two existing rigorous results on stable manifolds for orbits of the Kasner map in Bianchi IX: Béguin showed the existence of C^1- stable- manifolds for aperiodic orbits of the Kasner map ([**2**]), while Liebscher and co-authors ([**27, 28**]) showed the existence of Lipschitz-stable-manifolds for arbitrary orbits of the Kasner map not accumulating at one of the Taub points (Béguin also had to demand the latter condition).

Our result significantly extends Béguins results, who had to exclude all orbits that are perioidic or accumulate on any periodic orbit, a limitation which we were able to overcome. The techniques by Liebscher et al are able to treat both periodic and aperiodic chains, but yielded only Lipschitz-manifolds, i.e. the leaves of the foliation have less regularity.

But be aware that even though the stable manifolds constructed by Béguin and ourselves are C^1, this concerns only the regularity of the leaves of the foliation, and not the dependence on the base point. We do not get a C^1-foliation which would mean a C^1-dependece on the base point, but only a C^0-dependence of the (C^1-)leaves in the C^1-topology.

These aspects play a crucuial role when discussing the genericity of the foliation-results in BIX, i.e. how generic the set of initial conditions is both "down on the Kasner circle", as well as in the full space of trajectories. This involves delicate distinctions between topological vs. measure-theoretic genericity, and is subject of current research (for partial results, see [**38**][4]).

[4] in [**38**] it is shown that there are trajectories converging to every formal sequence given by a Kasner parameter u with at most polynomially bounded continued fraction expansion. This covers a set of full measure on the Kasner circle, but this does not mean that the set of coresponding initial conditions in a neighborhood of the Kasner circle has full measure. The reason is that there are counterexamples, i.e. it is possible to construct foliations where a countable set of "leaves" is attached to a set of base points that has full measure in the base space.

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1. Symbolic Computations with Mathematica

1.1. Constant Continued Fraction Expansion. u=[a,a,...]

For u=[m,a,a,...] and m=1...a, AND a= 1
m= 1 alpha= 16 beta= 4 k1= -1 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 2
m= 1 alpha= 12 beta= 3 k1= 1 k2= 2 k3= -1
m= 2 alpha= 24 beta= 5 k1= -2 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 3
m= 1 alpha= 11 beta= 3 k1= 3 k2= 3 k3= -1
m= 2 alpha= 19 beta= 4 k1= 1 k2= 3 k3= -1
m= 3 alpha= 33 beta= 6 k1= -3 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 4
m= 1 alpha= 11 beta= 3 k1= 5 k2= 4 k3= -1
m= 2 alpha= 18 beta= 4 k1= 4 k2= 5 k3= -1
m= 3 alpha= 28 beta= 5 k1= 1 k2= 4 k3= -1
m= 4 alpha= 45 beta= 7 k1= -4 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 5
m= 1 alpha= 11 beta= 3 k1= 7 k2= 5 k3= -1
m= 2 alpha= 18 beta= 4 k1= 7 k2= 7 k3= -1
m= 3 alpha= 27 beta= 5 k1= 5 k2= 7 k3= -1
m= 4 alpha= 39 beta= 6 k1= 1 k2= 5 k3= -1
m= 5 alpha= 59 beta= 8 k1= -5 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 6
m= 1 alpha= 11 beta= 3 k1= 9 k2= 6 k3= -1
m= 2 alpha= 18 beta= 4 k1= 10 k2= 9 k3= -1
m= 3 alpha= 27 beta= 5 k1= 9 k2= 10 k3= -1
m= 4 alpha= 38 beta= 6 k1= 6 k2= 9 k3= -1
m= 5 alpha= 59 beta= 8 k1= 1 k2= 6 k3= -1
m= 6 alpha= 75 beta= 9 k1= -6 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 7
m= 1 alpha= 11 beta= 3 k1= 11 k2= 7 k3= -1
m= 2 alpha= 18 beta= 4 k1= 13 k2= 11 k3= -1
m= 3 alpha= 27 beta= 5 k1= 13 k2= 13 k3= -1
m= 4 alpha= 38 beta= 6 k1= 11 k2= 13 k3= -1

m= 5 alpha= 51 beta= 7 k1= 7 k2= 11 k3= -1
m= 6 alpha= 67 beta= 8 k1= 1 k2= 7 k3= -1
m= 7 alpha= 93 beta= 10 k1= -7 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 8
m= 1 alpha= 11 beta= 3 k1= 13 k2= 8 k3= -1
m= 2 alpha= 18 beta= 4 k1= 16 k2= 13 k3= -1
m= 3 alpha= 27 beta= 5 k1= 17 k2= 16 k3= -1
m= 4 alpha= 38 beta= 6 k1= 16 k2= 17 k3= -1
m= 5 alpha= 51 beta= 7 k1= 13 k2= 16 k3= -1
m= 6 alpha= 66 beta= 8 k1= 8 k2= 13 k3= -1
m= 7 alpha= 84 beta= 9 k1= 1 k2= 8 k3= -1
m= 8 alpha= 113 beta= 11 k1= -8 k2= 1 k3= -1

For u=[m,a,a,...] and m=1...a, AND a= 9
m= 1 alpha= 11 beta= 3 k1= 15 k2= 9 k3= -1
m= 2 alpha= 18 beta= 4 k1= 19 k2= 15 k3= -1
m= 3 alpha= 27 beta= 5 k1= 21 k2= 19 k3= -1
m= 4 alpha= 38 beta= 6 k1= 21 k2= 21 k3= -1
m= 5 alpha= 51 beta= 7 k1= 19 k2= 21 k3= -1
m= 6 alpha= 66 beta= 8 k1= 15 k2= 19 k3= -1
m= 7 alpha= 83 beta= 9 k1= 9 k2= 15 k3= -1
m= 8 alpha= 103 beta= 10 k1= 1 k2= 9 k3= -1
m= 9 alpha= 135 beta= 12 k1= -9 k2= 1 k3= -1

1.2. 2-Periodic Continued Fraction Expansion. u=[a,b,...]

Now use a= 2 and b= 3

For u=[m,a,b,a,b,...] and m=1...b
m= 1 alpha= 15 beta= 4 k1= 7 k2= 7 k3= -2
m= 2 alpha= 24 beta= 5 k1= 3 k2= 7 k3= -2
m= 3 alpha= 34 beta= 6 k1= -5 k2= 3 k3= -2

For u=[m,b,a,b,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= 2 k2= 5 k3= -3
m= 2 alpha= 19 beta= 4 k1= -7 k2= 2 k3= -3

Now use a= 3 and b= 5

For u=[m,a,b,a,b,...] and m=1...b
m= 1 alpha= 11 beta= 3 k1= 23 k2= 17 k3= -3
m= 2 alpha= 23 beta= 5 k1= 23 k2= 23 k3= -3
m= 3 alpha= 33 beta= 6 k1= 17 k2= 23 k3= -3
m= 4 alpha= 46 beta= 7 k1= 5 k2= 17 k3= -3
m= 5 alpha= 60 beta= 8 k1= -13 k2= 5 k3= -3

For u=[m,b,a,b,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= 13 k2= 13 k3= -5
m= 2 alpha= 18 beta= 4 k1= 3 k2= 13 k3= -5
m= 3 alpha= 27 beta= 5 k1= -17 k2= 3 k3= -5

Now use a= 1 and b= 2

For u=[m,a,b,a,b,...] and m=1...b
m= 1 alpha= 16 beta= 4 k1= 2 k2= 3 k3= -1
m= 2 alpha= 25 beta= 5 k1= -1 k2= 2 k3= -1

For u=[m,b,a,b,a,...] and m=1...a
m= 1 alpha= 12 beta= 3 k1= -3 k2= 1 k3= -2

Now use a= 2 and b= 4

For u=[m,a,b,a,b,...] and m=1...b
m= 1 alpha= 15 beta= 4 k1= 12 k2= 10 k3= -2
m= 2 alpha= 24 beta= 5 k1= 10 k2= 12 k3= -2
m= 3 alpha= 34 beta= 6 k1= 4 k2= 10 k3= -2

m= 4 alpha= 47 beta= 7 k1= -6 k2= 4 k3= -2

For u=[m,b,a,b,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= 2 k2= 6 k3= -4
m= 2 alpha= 18 beta= 4 k1= -10 k2= 2 k3= -4

1.3. 3-Periodic Continued Fraction Expansion. u=[a,b,c,...]

Use a= 1 and b= 1 and c=2

For u=[m,b,c,a,...] and m=1...a
m= 1 alpha= 16 beta= 4 k1= 5 k2= -2 k3= 3

For u=[m,c,a,b,...] and m=1...b
m= 1 alpha= 12 beta= 3 k1= 2 k2= -3 k3= 3

For u=[m,a,b,c,...] and m=1...c
m= 1 alpha= 16 beta= 4 k1= -3 k2= -5 k3= 2
m= 2 alpha= 24 beta= 5 k1= 3 k2= -3 k3= 2

Use a= 3 and b= 3 and c=2

For u=[m,b,c,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= -23 k2= -22 k3= 7
m= 2 alpha= 19 beta= 4 k1= -10 k2= -23 k3= 7
m= 3 alpha= 33 beta= 6 k1= 17 k2= -10 k3= 7

For u=[m,c,a,b,...] and m=1...b
m= 1 alpha= 15 beta= 4 k1= -22 k2= -23 k3= 7
m= 2 alpha= 24 beta= 5 k1= -7 k2= -22 k3= 7
m= 3 alpha= 34 beta= 6 k1= 22 k2= -7 k3= 7

For u=[m,a,b,c,...] and m=1...c
m= 1 alpha= 11 beta= 3 k1= -7 k2= -17 k3= 10
m= 2 alpha= 23 beta= 5 k1= 23 k2= -7 k3= 10

Use a=b=c=1 (consistency check):

For u=[m,b,c,a,...] and m=1...a
m= 1 alpha= 16 beta= 4 k1= 2 k2= -2 k3= 2

For u=[m,c,a,b,...] and m=1...b
m= 1 alpha= 16 beta= 4 k1= 2 k2= -2 k3= 2

For u=[m,a,b,c,...] and m=1...c
m= 1 alpha= 16 beta= 4 k1= 2 k2= -2 k3= 2

Use a=b=c=3 (consistency check):

For u=[m,b,c,a,...] and m=1...a
m= 1 alpha= 11 beta= 3 k1= -30 k2= -30 k3= 10
m= 2 alpha= 19 beta= 4 k1= -10 k2= -30 k3= 10
m= 3 alpha= 33 beta= 6 k1= 30 k2= -10 k3= 10

For u=[m,c,a,b,...] and m=1...b
m= 1 alpha= 11 beta= 3 k1= -30 k2= -30 k3= 10
m= 2 alpha= 19 beta= 4 k1= -10 k2= -30 k3= 10
m= 3 alpha= 33 beta= 6 k1= 30 k2= -10 k3= 10

For u=[m,a,b,c,...] and m=1...c
m= 1 alpha= 11 beta= 3 k1= -30 k2= -30 k3= 10
m= 2 alpha= 19 beta= 4 k1= -10 k2= -30 k3= 10
m= 3 alpha= 33 beta= 6 k1= 30 k2= -10 k3= 10

1.4. Pre-Periodic Sequences. u=[m,b,a,b,a,...] and u=[m,a,b,c,...]

Now use a= 3, b= 2 and M= 5 for u=[m,b,a,b,a,...], m=1...M

m= 1 alpha= 15 beta= 4 k1= 7 k2= 7 k3= -2
m= 2 alpha= 24 beta= 5 k1= 3 k2= 7 k3= -2
m= 3 alpha= 34 beta= 6 k1= -5 k2= 3 k3= -2
m= 4 alpha= 47 beta= 7 k1= -17 k2= -5 k3= -2
m= 5 alpha= 61 beta= 8 k1= -33 k2= -17 k3= -2

Now use a= 1, b= 1 and c= 2 and M= 3 for u=[m,a,b,c,...], m=1...M

m= 1 alpha= 16 beta= 4 k1= -3 k2= -5 k3= 2
m= 2 alpha= 24 beta= 5 k1= 3 k2= -3 k3= 2
m= 3 alpha= 35 beta= 6 k1= 13 k2= 3 k3= 2

We can summarize our results from Mathematica as follows, where the smoothness of the coordinate change is set to one ($k = 1$ in the Takens Theorem):

- for constant contiued fraction expansions the conditions are violated in the cases $u = [m, a, a, ...]$ for $a = 1...9$, so there is no simple infinite periodic heteroclinic chain with constant continued fraction development. We see, for example, in the case $u = [1, 1, ...]$ of the 3-cycle, it holds that $(k_1, k_2, k_3) = (-1, 1, -1)$, which means that $\lambda_2 = \lambda_1 + \lambda_3$, which can be checked directly and serves as a consistency check.
- for 2-periodic continued fractions like $u = [2, 3, 2, 3, ...]$ or $u = [3, 5, 3, 5, ...]$, the Resonance Sign Condition (RSC) is violated, i.e. Takens-Linearization is possible. But note that for this argument to work, we have to require the coefficients to be greater than one, even after cancelling out a possible common factor. This is illustrated by the examples $u = [1, 2, 1, 2, ...]$ and $u = [2, 4, 2, 4, ...]$.
- For $u = [1, 1, 2, 1, 1, 2, ...]$, the RSC is also violated, illustrating the fact that we don't have to require the coefficients to be greater than one if the period is greater than two. For $u = [3, 3, 2, 3, 3, 2, ...]$, the RSC is also violated. However, even without using this fact, the chain would qualify for Takens Linearization, as the sum of the order of the resonances is always greater than the required α at all base points.
- We have also included the examples for $u = [a, b, c, a, b, c, ...]$ with $a = b = c = 1$ and $a = b = c = 3$ as consistency check: the formulas remain correct, but due to a common factor in the resulting coefficients, there is an earlier resonance that we already found in the section on constant contined fraction expansions.
- the 1-pre-periodic sequences $u = [3, 1, 1, 2, 1, 1, 2, ...]$ and $u = [5, 3, 2, 3, 2, ...]$ show that if the first coefficient m is bigger than the ones that follow, it cannot be assured that the NRC's are met: In the first case, this fails for $m = 3$, in the second case for $m = 4$ and $m = 5$, which means that Takens Linearization is not possible.